International Journal of Theoretical Physics, Vol. 31, No. 2, 1992

Einstein Algebras and General Relativity

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Received May 27, 1991

A purely algebraic structure called an Einstein algebra is defined in such a way that every spacetime satisfying Einstein's equations is an Einstein algebra but not vice versa. The Gelfand representation of Einstein algebras is defined, and two of its subrepresentations are discussed. One of them is equivalent to the global formulation of the standard theory of general relativity; the other one leads to a more general theory of gravitation which, in particular, includes so-called regular singularities. In order to include other types of singularities one must change to sheaves of Einstein algebras. They are defined and briefly discussed. As a test of the proposed method, the sheaf of Einstein algebras corresponding to the spacetime of a straight cosmic string with quasiregular singularity is constructed.

INTRODUCTION

The idea of presenting general relativity as a special case of a more general purely algebraic structure belongs to Geroch (1972). He defined an Einstein algebra as consisting of a commutative ring with a subring isomorphic to real numbers, and a metric such that the contraction property is satisfied and the Ricci tensor vanishes (or, alternatively, such that Einstein's equations with suitable sources are satisfied). Geroch did not go far beyond showing that the collection of all real-valued functions on a spacetime manifold satisfying Einstein's equations is an Einstein algebra, and making a few comments concerning the role of spacetime events in general relativity and quantum theories; he also pointed out how experimental predictions could eventually be derived from the formalism of Einstein algebras.

In the present work, after giving necessary algebraic preliminaries (Section 1), mainly to establish notation and conventions, the Geroch program is developed (Section 2). Stress is put on functional representations of Einstein algebras. Two such representations are discussed in some detail. One of them

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0020-7748/92/0200-0277$06.50/0

is equivalent to the global formulation of orthodox general relativity; the other one is its generalization. In particular some mild types of singularities become intrinsic elements of the new theory. It turns out that in order to include other types of singularities (also curvature singularities) into the theoretical scheme, one must change from Einstein algebras to sheaves of Einstein algebras (Section 3).

The method is tested on the example of spacetime due to a straight cosmic string with quasiregular singularity. The sheaf of Einstein algebras fully modeling such a configuration is constructed (Section 4). Finally, in Section 5, some comments are given with emphasis on the gravity quantization problem and observational possibilities of algebraic theories of gravitation.

1. ALGEBRAIC PRELIMINARIES

The basic object of our study is a *linear algebra*, i.e., a commutative ring C, with respect to two internal operations: addition and multiplication, together with one external operation $\mathbb{K} \times C \to C$ defined by $(\rho, x) \mapsto \rho x$, $\rho \in \mathbb{K}$, $x \in C$, such that the set C is a linear space with respect to the internal addition and the external multiplication.[2]

A module $\mathbb{W}$ over the ring C (a C-module) is called a *linear-C-module*. Every linear C-module $\mathbb{W}$ with identity is an $\mathbb{R}$-module. Indeed, if $\rho \in \mathbb{R}$, and 1 is the identity of C, one may identify $\rho \cdot 1$ with ρ, which gives $\mathbb{R} \subset C$, and consequently the multiplication $\rho \cdot v$, $v \in \mathbb{W}$, is also defined.

In the following, only linear algebras and modules are considered; the word "linear" is often omitted.

Let C be an algebra. Any linear mapping $V: C \to C$ such that

$$V(\alpha\beta) = V(\alpha)\beta + \alpha V(\beta)$$

for any $\alpha, \beta \in C$, is said to be a *C-vector* (or simply a *vector*). The set of all C-vectors is denoted by $\mathscr{X}(C)$. To denote the value $V(\alpha)$ of a C-vector at $\alpha \in C$ we also use the symbol $\partial_v \alpha$, and we shall say that $\partial_v \alpha$ is a *derivative of α in the direction V*. It can be easily shown that $\mathscr{X}(C)$ is a Lie algebra with respect to the commutator $[X, Y] = X \circ Y - Y \circ X$, $X, Y \in \mathscr{X}(C)$.

The set of C-linear mappings (or homomorphisms) of the C-module $\mathscr{X}(C)$ into C defines the dual module $\mathscr{X}(C)^*$; we write

$$\mathscr{X}(C)^* = \mathscr{L}_c(\mathscr{X}(C); C)$$

Every C-linear mapping $W \in \mathscr{X}(C)^*$ is called a *C-covector*.

[2]A standard reference for this section is Sikorski (1972), Chapter 5.

Let $\mathbb{W}_1, \ldots, \mathbb{W}_n, \mathbb{W}$ be C-modules. Any C-n-linear mapping

$$T\colon \quad \mathbb{W}_1, \ldots, \mathbb{W}_n \to \mathbb{W}$$

is called a *C-n-tensor* (or *tensor*, for brevity). The set of all such tensors is denoted by $\mathscr{L}_c(\mathbb{W}_1, \ldots, \mathbb{W}_n; \mathbb{W})$. If $\mathbb{W} = C$, the tensor is called the *scalar tensor*. The C-module $\mathscr{L}_c(\mathbb{W}; C)$ of C-linear mappings $L\colon \mathbb{W} \to C$ is called the *dual* of $\mathbb{W}$ and is denoted by $\mathbb{W}^*$. With the help of $\mathbb{W}$ and $\mathbb{W}^*$ one can construct C-n-tensors of various valences (by defining corresponding tensor products).

Let $\mathbb{W}$ be a C-module and $g\colon \mathbb{W} \times \mathbb{W} \to C$ a 2-C-linear mapping, and let us choose any vector $V \in \mathbb{W}$. The formula

$$V_g(W) = g(V, W)$$

for every $W \in \mathbb{W}$, defines uniquely a covector $V_g \in \mathbb{W}^*$. We define a C-linear mapping γ by

$$[\gamma(V)](W) = g(V, W)$$

g is said to be a *nondegenerate* mapping if for every $U \in \mathbb{W}^*$ there exists exactly one $V \in \mathbb{W}$ such that $U = \gamma(V)$. In such a case, there exists a one-to-one mapping of the C-module $\mathbb{W}$ onto the *C-module* $\mathbb{W}^*$. g is said to be *symmetric* if $g(V, W) = g(W, V)$, for every $V, W \in \mathbb{W}$. A nondegenerate and symmetric 2-C-linear mapping $g\colon \mathbb{W} \times \mathbb{W} \to C$ is called a *scalar product* (in the module $\mathbb{W}$).

One can show that if there exists a scalar product g in $\mathbb{W}$, then $\mathbb{W}$ is reflexive, and $g^*(X, Y) := g(\gamma^{-1}(X), \gamma^{-1}(Y))$, for $X, Y \in \mathbb{W}^*$, is a scalar product in $\mathbb{W}^*$.

Let us suppose that $\mathbb{W}$ has a basis $(W_0, W_1, \ldots, W_m)$ and g is a particular scalar product in $\mathbb{W}$. We shall say that this basis, or this scalar product, is *pseudoorthonormal* (or *Lorentz*) if

$$g(W_i, W_j) = \eta_{ij}$$

with

$$\eta_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j \\ -1 & \text{if} \quad i = j = 0 \\ 1 & \text{if} \quad i = j \neq 0 \end{cases}$$

where 0, -1, and 1 are to be understood as constant functions belonging to C. A C-module in which there exists such a basis will be said a *pseudo-orthonormal* or *Lorentz* C-module.

The *covariant derivative in the C-differential module* $\mathbb{W}$ is any mapping

$$\nabla \in \mathscr{L}_C(\mathscr{X}(C); \mathscr{L}_{\mathbb{R}}(\mathbb{W}; \mathbb{W}))$$

satisfying the condition

$$\nabla_V(\alpha W) = \partial_V \alpha \cdot W + \alpha \nabla_V W$$

where $V \in \mathscr{X}(C)$, $W \in \mathbb{W}$, $\alpha \in C$, and ∇_V: $\mathbb{W} \to \mathbb{W}$ is a function assigning $\nabla_V W$ (called the *directional derivative* of W in the direction V) to every $W \in \mathbb{W}$. It can be demonstrated that for any scalar product G in the C-module $\mathscr{X}(C)$ there exists exactly one symmetric covariant derivative ∇ in $\mathscr{X}(C)$ such that $\nabla G = 0$.

Let $\mathbb{W}$ be a C-module, and ∇ a covariant derivative in $\mathbb{W}$. By definition, ∇ is a mapping which assigns to every $X \in \mathscr{X}(C)$ another mapping, namely

$$\nabla: \quad \mathbb{W} \to \mathscr{L}_{\mathbb{R}}(\mathbb{W}; \mathbb{W})$$

In general, this transformation does not preserve Lie brackets, i.e.,

$$R_{XY} \equiv [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \neq 0$$

for X, $Y \in \mathscr{X}(C)$. One can see that R_{XY}: $\mathbb{W} \to \mathbb{W}$ is a mapping such that $W \mapsto R_{XY}W$, for every $W \in \mathbb{W}$.

It can be shown that $R_{XY}W$, viewed as a function of three variables—X, $Y \in \mathscr{X}(C)$, $W \in \mathbb{W}$—is a tensor, i.e., $R_{XY} \in \mathscr{L}_c(\mathbb{W}; \mathbb{W})$.

The mapping which to any two vectors X, $Y \in \mathscr{X}(C)$ assigns the mapping R_{XY}: $\mathbb{W} \to \mathbb{W}$ is denoted by R; therefore

$$R \in \mathscr{L}_c(\mathscr{X}(C), \mathscr{X}(C); \mathscr{L}_c(\mathbb{W}, \mathbb{W}))$$

and R is called the *Riemann tensor* or *curvature tensor of the covariant derivative* ∇.

Let us consider a fixed $W \in \mathbb{W}$. The mapping which to any two elements X, $Y \in \mathscr{X}(C)$ assigns $R_{XY}W \in \mathbb{W}$ is denoted by RW; therefore

$$RW \in \mathscr{L}_c(\mathscr{X}(C), \mathscr{X}(C); \mathbb{W})$$

The multilinear mapping

$$\hat{R}: \quad \mathscr{X}(C) \times \mathscr{X}(C) \times \mathbb{W} \times \mathbb{W}^* \to C$$

defined by

$$\hat{R}(X, Y, W, U) = U(R_{XY}W), \qquad X, Y \in \mathscr{X}(C), \quad W \in \mathbb{W}, \quad U \in \mathbb{W}^*$$

is called the *scalar curvature* (or *Riemann*) *tensor*. It can be shown that if the module $\mathbb{W}$ is reflexive, i.e., if $\mathbb{W}^{**} = \mathbb{W}$, the scalar curvature tensor $\hat{R}$ uniquely determines the curvature tensor R. For the purposes of the present paper it would be enough to introduce the *scalar* curvature tensor; notice, however, that the above machinery is worthwhile to consider since it suggests further generalizations of the concept of Einstein algebras.

Let G denote a certain fixed two-covariant symmetric tensor on the C-module $\mathbb{W}$, i.e., $G \in \mathscr{L}_c(\mathbb{W}, \mathbb{W}; C)$. We define the C-linear mapping

$$GR: \quad \mathscr{X}(C) \times \mathscr{X}(C) \times \mathbb{W} \times \mathbb{W} \to C$$

$$GR(X, Y, U, W) = G(R_{XY}U, W)$$

for $X, Y \in \mathscr{X}(C)$, $U, W \in \mathbb{W}$. The tensor $GR \in \mathscr{L}_c(\mathscr{X}(C), \mathscr{X}(C), \mathbb{W}, \mathbb{W}; C)$ is called the *covariant curvature tensor*. It is uniquely determined by the covariant derivative ∇ and the tensor G. In the following we shall assume that G is a metric tensor such that $\nabla G = 0$.

Let $\mathbb{W}$ be a C-module with a basis $W_1, \ldots, W_n$, and $\mathbb{W}^*$ the dual C-module with the dual basis $W^1, \ldots, W^n$. Let us also consider a C-linear mapping $L: \mathbb{W} \to \mathbb{W}$. The *trace* of L, $\operatorname{tr} L$, is defined to be

$$\operatorname{tr} L = W^i(LW_i) \in C, \qquad i = 1, \ldots, n$$

This definition is independent of the choice of a particular basis. From this definition it follows that

$$\operatorname{tr} \in \mathscr{L}_c(\mathscr{L}_c(\mathbb{W}; \mathbb{W}); C)$$

Let us consider a C-module $\mathbb{W}$ having a basis $(V_1, \ldots, V_n)$; in such a case, the trace $\operatorname{tr} L \in C$ of any tensor $L \in \mathscr{L}_c(\mathbb{W}; \mathbb{W})$ is determined, $\operatorname{tr} \in \mathscr{L}_c(\mathscr{L}_c(\mathbb{W}; \mathbb{W}); C)$. Let us also assume that ∇ is a covariant derivative in $\mathbb{W}$ and that $\mathbb{W} = \mathscr{X}(C)$. If R is the curvature tensor of ∇, the tensor

$$\operatorname{Ric} \in \mathscr{L}_c(\mathscr{X}(C), \mathscr{X}(C); C)$$

defined by $\operatorname{Ric}(Y, Z) = \operatorname{tr}_X(R_{XY}Z)$, $Y, Z \in \mathscr{X}(C)$, for any $X \in \mathscr{X}(C)$, is said to be the *Ricci tensor*.

$R_{XY}Z$, with Y and Z fixed, should be thought of as a linear function of the variable X transforming the module $\mathscr{X}(C)$ into itself. $\operatorname{Ric}(Y, Z)$ is the trace of this function.

2. EINSTEIN ALGEBRAS AND THEIR REPRESENTATIONS

By an *Einstein algebra* $\mathscr{A}$ we mean a linear algebra C satisfying the following conditions:[3]

(i) The C-module $\mathbb{W} = \mathscr{X}(C)$ of all C-vectors is the Lorentz C-module.

(ii) There exists a covariant derivative ∇ in $\mathbb{W}$ such that $\nabla g = 0$, where g is the Lorentz scalar product in $\mathbb{W}$.

[3]Alternatively, one could assume C to be a commutative ring having a subring $\mathscr{R}$ isomorphic with the real numbers such that the identity of $\mathscr{R}$ is the identity of C (Geroch, 1972).

(iii) $\mathrm{Ric}=0$.

$\mathscr{A}$ is an *extended Einstein algebra* if, instead of (iii), the following condition is satisfied:

(iii′) $\mathrm{Ein}+\Lambda g=T$, where Ein is the Einstein tensor, Λ is the cosmological constant, and T is a suitable energy-momentum tensor.

Every spacetime manifold satisfying Einstein's equations is an (extended) Einstein algebra, but not vice versa. Einstein algebras are not only purely algebraic structures, but they are also generalizations of the general theory of relativity.

Let $\mathscr{A}^*$ be the dual of $\mathscr{A}$ as a vector space over $\mathbb{K}$. By $\mathscr{A}^\wedge \subset \mathscr{A}^*$ we denote the "algebraic dual" of $\mathscr{A}$, i.e., the set of all homomorphisms $\{\phi: \mathscr{A} \to \mathbb{K}\}$. There is a bijection between $\mathscr{A}^\wedge$ and Spec $\mathscr{A}$, the set of all strictly maximal ideals in $\mathscr{A}$. A representation of $\mathscr{A}$, $\rho: \mathscr{A} \to \mathbb{K}^{\mathscr{A}^\wedge}$, given by $\rho(x)(\phi)=\phi(x)$, $x\in\mathscr{A}$, $\phi\in\mathscr{A}^\wedge$, is the *Gelfand representation* of an Einstein algebra $\mathscr{A}$. It is a *universal* representation of $\mathscr{A}$ in the sense that every representation of $\mathscr{A}$ is equivalent to a subrepresentation of ρ. It is also a *natural* representation of $\mathscr{A}$ in the sense of category theory (Palais, 1981).

We define a *structural ring* (over $\mathbb{K}$) of a set M to be a subalgebra $\mathscr{C}$ of the algebra $\mathbb{K}^M$ of $\mathbb{K}$-valued functions on M which separate points in M. A *ringed space* (over $\mathbb{K}$) is a pair $(M, \mathscr{C})$, where M is any set and $\mathscr{C}$ is a structural ring on M.

Let us suppose that the Gelfand representation of an Einstein algebra $\mathscr{A}$ separates points in $\mathscr{A}^\wedge$ (this is not a limitation since, if necessary, we can always define a suitable equivalence relation which would do the job). Of course, $(\mathscr{A}^\wedge, \rho(\mathscr{A}))$ is a ringed space; we call it an *Einstein ringed space*.

Let $\mathscr{A}$ be an Einstein algebra, $M\subset\mathscr{A}^\wedge$, and let $C^\infty(M)\subset\mathbb{R}^M$ denote the set of all smooth real functions on M. $\kappa: \mathscr{A} \to C^\infty(M)$ is a subrepresentation of the Gelfand representation of $\mathscr{A}$; it is called a *Geroch representation* of $\mathscr{A}$ (Geroch, 1972). A ringed space $(M, C^\infty(M))$, called a *Geroch ringed space*, is a smooth manifold satisfying Einstein's field equations. The Geroch representation of $\mathscr{A}$ is equivalent to the orthodox theory of general relativity (but it is global from the very beginning).

Now, let $C\subset\mathbb{R}^M$ denote the set of all real functions on the set M (we endow M with the weakest topology τ_c in which functions of C are continuous) and satisfying the following axioms: (1) C is closed with respect to localization, and (2) C is closed with respect to superposition with smooth functions on the Euclidean space.

A function f, defined on $A\subset M$, is said to be a *local C-function* if, for every $p\in A$, there is a neighborhood B of p in the topological space (A, τ_A), where τ_A is the topology induced in A by τ_c, and a function $g\in C$ such that $g|B=f|B$. The set of all local C-functions is denoted by C_A. One obviously has $C\subset C_M$. If $C=C_M$, the family C is said to be *closed with respect to localization*.

Let C be a family of real functions on M. It is said to be *closed with respect to superposition with smooth Euclidean functions* if for any $n \in \mathbb{N}$ and any function $\omega \in C^\infty(\mathbb{R}^n)$, $f_1, \ldots, f_n \in C$ implies $\omega \circ (f_1, \ldots, f_n) \in C$.

A family C of real functions on M satisfying conditions (1) and (2) above is called a *differential structure* on M, and it is treated, *ex definitione*, as the family of smooth functions on M. A pair (M, C), where C is a differential structure on M, is called a *differential space* [*in the sense of Sikorski* (1967, 1971, 1972)]. On the strength of condition (2), C is a linear algebra, and consequently any differential space (M, C) is also a ringed space; it is called a *Sikorski ringed space*; correspondingly, we also speak of a *Sikorski representation* of an Einstein algebra $\mathscr{A}$, $\sigma: \mathscr{A} \to C$ [a detailed proof that all Einstein algebra axioms are satisfied can be found in Heller *et al.* (1989); on algebraic foundations of the theory of differential spaces see Heller (1991)].

Sikorski representations of Einstein algebras correspond to a theory which is more general than the usual theory of general relativity. The differential space axioms do not demand that it should be locally diffeomorphic to $\mathbb{R}^n$ (if we add this postulate, the differential space changes into a smooth manifold). Therefore, Sikorski representations of Einstein algebras cover a set of possibilities which are automatically excluded from general relativity. Although conditions (i) and (iii) of the Einstein algebra definition are very restrictive, Sikorski representations of Einstein algebras are flexible enough to cover nonsmooth situations which in general relativity would be considered as true spacetime singularities. The Lorentz metric, curvature tensor, and Ricci tensor (and consequently Einstein's equations) can be defined on a differential space (M, C), even if it is not a manifold, provided there is an open covering $\mathscr{B}$ of M such that on every open set $B \in \mathscr{B}$ there exist n smooth tangent vector fields forming a vector basis (i.e., a vector basis in the C-module of all smooth tangent vector fields on M; "smooth" is here understood in the sense of the theory of differential spaces: a function f is smooth if $f \in C$; see, for instance, Gruszczak *et al.*, 1988). In such a case one says that (M, C) is of constant *differential dimension n* [on the differential dimension of differential spaces see Heller *et al.* (1991)]. Such a generalized theory of general relativity has been considered in Gruszczak *et al.* (1988, 1989) and Heller *et al.* (1989). In particular, the so-called regular singularities [which essentially originate by cutting off some parts of spacetime (Ellis and Schmidt, 1977)] are "intrinsic elements" of the theory (Heller and Sasin, 1991). It turns out, however, that stronger types of singularities (including curvature singularities) prevent the corresponding differential space from being of constant differential dimension, and consequently cannot be treated as "intrinsic elements" of the Einstein algebra. To allow stronger types of singularities to become parts of the theory, we should change from Einstein algebras to sheaves of Einstein algebras.

3. SHEAVES OF EINSTEIN ALGEBRAS

Let $\bar{M}$ be any (nonempty) set equipped with any topology top $\bar{M}$. In view of future applications to general relativity, we shall additionally assume that $\bar{M} = M \cup \partial M$, and M is open and dense in $\bar{M}$; ∂M is the *boundary of* M. Since (for the time being) top $\bar{M}$ is any topology on $\bar{M}$, the last condition is rather a mild limitation of generality.

By a *sheaf of Einstein algebras* we understand a sheaf $\mathscr{C}$ of linear function algebras over the topological space ($\bar{M}$, top $\bar{M}$) such that for any $p \in M$ and any $U = \text{top } M$, $p \in U$, where top M is the topology on M induced from that of $\bar{M}$, $\mathscr{C}(U)$ is an Einstein algebra.

Since $M \in \text{top } \bar{M}$, the above definition implies that $\mathscr{C}(M)$ is an Einstein algebra. In particular, the sheaf $\mathscr{C}_M = \mathscr{C}|M$ is locally free, i.e., the $\mathscr{C}(M)$-module $\mathscr{X}(\mathscr{C}_M)$ of cross sections of the sheaf $\mathscr{C}_M$ has a $\mathscr{C}(M)$-basis, and the Lorentz scalar product g exists in $\mathscr{C}_M$. In fact, $(M, \mathscr{C}_M)$ is a ringed space.

Following Hochschild (1965), the pair $(M, \mathscr{O})$, whre M is a topological space and $\mathscr{O}$ a sheaf of function algebras (a *functional structure*) on M, is called the (*functional*) *structured space*. If $\mathscr{O}$ is a sheaf of Einstein algebras, $(M, \mathscr{O})$ is called *Einstein structured space*.

Every differential manifold M can be regarded as a structured space $(M, \mathscr{O})$ with the functional structure given by the sheaf $\mathscr{O}$ of germs of smooth real functions on M. If M is a spacetime manifold of general relativity, then the corresponding $(M, \mathscr{O})$ is the Einstein structured space. Evidently, there are many Einstein structured spaces which are not smooth manifolds.

Let (M, C) be a differential space (in the sense of Sikorski), and let us consider the topological space (M, τ_c), where τ_c is the weakest topology on M in which functions of C are continuous. The family $C(M)$ of C-functions on (M, τ_c) is a sheaf of function algebras such that $C(U) = C_U$, for $U \in \tau_c$, C_U being the set of local C-functions on U. Consequently, $(M, C(M))$ is a structured space. If $C(U)$, for all $U \in \tau_c$, is an Einstein algebra, then $(M, C(M))$ is the Einstein structured space.

4. AN EXAMPLE: A SPACETIME WITH SINGULARITY

The metric

$$ds^2 = -dt^2 + dr^2 + r^2\, d\theta^2 + dz^2$$

where t, $z \in (-\infty, \infty)$, $r \in (0, \infty)$, $\theta \in \langle 0, 2\pi - \Delta)$, and $\Delta \in (0, 2\pi)$, first discovered by Staruszkiewicz (1963) in his 3-dimensional general relativity, can be interpreted as describing the external gravitational field of a straight cosmic string (see also Vilenkin, 1981; Hiscock, 1985; Gott, 1985). This spacetime was analyzed in terms of Sikorski's theory of differential spaces by Gruszczak *et al.* (1991).

Let us define a 4-dimensional "conic" hypersurface embedded in $\mathbb{R}^5$,

$$C^{(4)} = \{ p \in \mathbb{R}^5 \colon [(z^1)^2 + (z^2)^2]^{1/2} = az^4 \}$$

where $p = (z^0, z^1, z^2, z^3, z^4)$, $a \in \mathbb{R}$. The set of singular points (conic singularity) of this hypersurface has the form

$$S = \{ p \in \mathbb{R}^5 \colon p = (z^0, 0, 0, z^3, 0), z^0, z^3 \in \mathbb{R} \}$$

Let C_0 be any set of real functions on a set M. There exists the smallest differential structure C (in the sense of Sikorski) on M such that $C_0 \subset C$ and $\tau_{c_0} = \tau_c$. In such a case C is said to be *generated* by C_0, and C_0 is called the *set of generators of* C; one writes $C = \text{Gen } C_0$. (M, C) is said to be *finitely generated* if the set of generators is finite. It can be shown that a function f belongs to C if and only if, for every point $p \in M$, there exist a neighborhood U of p, functions $\phi_1, \ldots, \phi_n \in C_0$, and a real function ω on $\mathbb{R}^n$ such that $f|U = \omega \circ (\phi_1, \ldots, \phi_n)|U$.

Now, we shall describe $C^{(4)}$ as a finitely generated d-space. Let $\tilde{P} := \mathbb{R}^2 \times \langle 0, \infty) \times \langle 0, 2\pi \rangle$ be a "parameter space," and $\alpha_i \colon \tilde{P} \to \mathbb{R}$, $i = 0, 1, \ldots, 4$, real functions parametrizing the hypersurface $C^{(4)}$ in the following way:

$$z^0 = \alpha_0(q) = t$$

$$z^1 = \alpha_1(q) = \rho \cos \phi$$

$$z^2 = \alpha_2(q) = \rho \sin \phi$$

$$z^3 = \alpha_3(q) = z$$

$$z^4 = \alpha_4(q) = a\rho$$

where $q = (t, z, \rho, \phi) \in \tilde{P}$, $r = \rho(a^2 + 1)^{1/2}$, and $\theta = \phi(a^2 + 1)^{-1/2}$.

Let $\tilde{\mathscr{P}}$ be the differential structure on $\tilde{P}$ generated by $\{\alpha_0, \alpha_1, \ldots, \alpha_4\}$, $\tilde{\mathscr{P}} = \text{Gen}\{\alpha_0, \alpha_1, \ldots, \alpha_4\}$. The differential space $(\tilde{P}, \tilde{\mathscr{P}})$ is not Hausdorff, since functions α_i, $i = 0, 1, \ldots, 4$, do not distinguish the points $(t, z, \rho, 0)$ and $(t, z, \rho, 2\pi)$. To cure this situation, let us define the Hausdorff equivalence relation ρ_H in the following way: for any $q_1, q_2 \in \tilde{P}$, $q_1 \, \rho_H \, q_2$ if and only if $\alpha_i(q_1) = \alpha_i(q_2)$, $i = 0, 1, \ldots, 4$. Let $\mathscr{P} := \tilde{\mathscr{P}}/\rho_H$ and $P := \tilde{P}/\rho_H$. We obtain the Hausdorff differential space $(P, \mathscr{P})$. It can be easily seen that $\mathscr{P} = \text{Gen}\{\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_4\}$, where

$$\hat{\alpha}_i([p]) := \alpha_i(p)$$

for $p \in \tilde{P}$, $[p] \in P$, $i = 0, 1, \ldots, 4$.

The differential space $(P, \mathscr{P})$ models the spacetime of a cosmic string with singularity. In fact, $(P, \mathscr{P})$ is diffeomorphic to the differential space $(C^{(4)}, (\mathscr{E}_5)_{C^{(4)}})$ which is a differential subspace of $(\mathbb{R}^5, \mathscr{E}_5)$, where $\mathscr{E}_5$ denotes the natural differential structure on $\mathbb{R}^5$. Indeed, the mapping

$$\hat{F} := (\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_4): \quad P \to \mathbb{R}^5$$

is a diffeomorphism of $(P, \mathscr{P})$ onto the image $(\hat{F}(P), (\mathscr{E}_5)_{\hat{F}(P)})$, and by direct computation one can check that $\hat{F}(P) = C^{(4)}$ [for details see Gruszczak *et al.* (1991)].

Points of the form $[(t, z, 0, \phi)] \in P$, where $t, z \in \mathbb{R}$, $\phi \in \langle 0, 2\pi \rangle$, represent the singularity since $\hat{F}[(t, z, 0, \phi)] \in S$. Let us denote the set of all singular points by ∂P, and let $\hat{P} = P - \partial P$. Now, $\hat{F}|_{\hat{P}}$ is a diffeomorphism of the differential subspace $(\hat{P}, \mathscr{P}_{\hat{P}})$ onto $(C^{(4)} - S, (\mathscr{E}_5)_{C^{(4)} - S})$, which is the spacetime manifold of the cosmic string.

$\mathscr{P}$ is a sheaf of linear function algebras on the topological space $(P, \tau_{\mathscr{P}})$, where $P = \hat{P} \cup \partial P$, and, as can be easily seen, $\hat{P}$ is open and dense in P. $\mathscr{X}(\mathscr{P}(\hat{P}))$ is a locally free $\mathscr{P}(\hat{P})$-module (and even a Lorentz module), and $\mathrm{Ric} = 0$ is satisfied on $(\hat{P}, \mathscr{P}_{\hat{P}})$. Consequently, $(P, \mathscr{P})$ is an Einstein structured space.

The spacetime considered here of a cosmic string together with its quasi-regular singularity has been recently investigated with the help of standard methods by Vickers (1985, 1987, 1990).

5. DISCUSSION

Einstein algebras have various functional representations (subrepresentations of the Gelfand representation), one of which, the Geroch representation, is equivalent to the global version of standard general relativity.

It turns out that Einstein algebras can be naturally organized into sheaves of Einstein algebras. The only deviation form full generality in our construction is the assumption that the sheaf is defined on a topological space $\bar{M} = M \cup \partial M$ such that M is open and dense in $\bar{M}$. Taking into account the fact that the topology in $\bar{M}$ is not *a priori* specified, this assumption is not very restrictive, but it allows one to consider spacetime singularities as the topological boundary ∂M of spacetime M.

Notice, however, that neither spacetime itself nor its boundary are primitive elements of the theory. The main advantage of the purely algebraic treatment is that no spacetime events appear in it from the very beginning. There are elements of abstract algebras that should be considered as the primary "objects" of the theory. Only after changing to the Gelfand representation of the given Einstein algebra $\mathscr{A}$ (or to some of its subrepresentations) do these elements become real-valued functions on the set Spec $\mathscr{A}$ of strictly maximal ideals of $\mathscr{A}$ (or on some of its subsets) which assume the role of spacetime "events." In Geroch representation the set of such events is a smooth manifold (spacetime in the usual sense); in other representations (for instance, in the Sikorski representation) some "events" can be singular.

The fact that we are dealing with functional spaces (ringed spaces or structured spaces) makes the theory of Einstein algebras more similar to the formalism of quantum theories. The resemblance goes further. As is well known, standard quantum mechanics can be elegantly given the abstract structure of C^*-algebras. A C^*-algebra is a Banach algebra with the operation called involution (and denoted by $*$) which satisfies conditions analogous to those of the usual conjugation operation. On the strength of the Gelfand–Najmark theorem, every commutative C^*-algebra A is isomorphic with the algebra $C(M)$ of continuous complex functions on a compact set M. The isomorphism $A \to C(M)$ defines the Gelfand representation of A, and M is the set of maximal ideals of A (see, for instance, Maurin, 1980, pp. 673–678). The analogy with Einstein algebras and their Gelfand representation is striking; it certainly deserves further investigation.

Observational possibilities connected with Einstein algebras were briefly discussed by Geroch (1972). An essential difference between observational aspects of Einstein algebras (and even more of sheaves of Einstein algebras) and those of standard general relativity consists in the fact that in the Einstein algebras they should be computed in terms of global tensor fields rather than locally as is usually done in the majority of existing theories of gravity. Such computations, although tedious, are perfectly possible (Sikorski, 1972, Chapter 5). As we have seen, there exist representations of Einstein algebras which are more general than general relativity; in such circumstances new observational effects should not be a surprise. In particular, these versions of new theories which incorporate singularities into their own structures could lead to new observational effects for algebras (or sheaves of algebras) modeling physical situations in the presence of singularities (for instance, black holes or early stages of the universe's evolution).

ACKNOWLEDGMENT

I express my gratitude to Dr. W. Sasin for discussions and valuable suggestions. This work has been supported by KBN grant 20447/91-01.

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